Collisionless Flow of a Plasma Mixture

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The collisionless flow of a plasma consisting of two ion-species about a solid body is analyzed by extending the cold-ion theory of Lam and Greenblatt. A mixture Mach number is defined that governs the over-all behavior of the flow. The ion flux at the "ram position" of the solid body in a subsonic flow is computed, and the Prandtl-Meyer fan solution for a supersonic flow is presented.

I. Introduction

SCIENTIFIC satellites often carry instruments to sample the composition of the ionosphere. Generally, the instrument is a mass spectrometer that is flush-mounted on the satellite surface. The number fluxes of the various ions entering the orifice of the spectrometer are measured. Our problem here is to construct a theory for the analysis of these data in view of the fact that the ionospheric plasma is generally a mixture.

If the particles being collected by the spectrometer were neutral particles, the number fluxes f_i can readily be calculated by the well-known free molecular formula[1]

$$f_i = n_{i,\infty}(kT_i/m_i)^{1/2}F(S_i) \tag{1.1a}$$

$$F(S_i) = (2\pi)^{-1/2}\exp(-S_i^2/2) + S_i[1 + \operatorname{erf}(S_i^2/2)^{1/2}]/2 \tag{1.1b}$$

$$S_i = \frac{m_i}{kT_i}^{1/2}(\bar{U}_\infty\cdot\bar{\nu}), \quad \operatorname{erf}\lambda \equiv 2(\pi)^{-1/2}\int_0^\lambda e^{-x^2}\,d_\chi \tag{1.1c}$$

where $\bar{\nu}$ is the inward unit normal of a generally convex body, $\bar{U}_\infty$ is the freestream velocity. If S_i is very large, Eq. (1.1) reduces to

$$f_i(S_i \to \infty) = n_i(\bar{U}_\infty\cdot\bar{\nu}), \quad (\bar{U}_\infty\cdot\bar{\nu}) > 0(\text{only}) \tag{1.2}$$

which is often referred to as the "scoop formula." Equations (1.1) and (1.2) are relatively simple to apply, but clearly they are not valid when we deal with charged particles.

When the freestream plasma consists of only a single ion species, Lam and Greenblatt[2,3] had constructed an approximate theory for the calculation of the flowfield and the ion number flux. It is the purpose of this paper to construct an extension of this theory for a flowing plasma mixture.

II. General Formulation

For simplicity, we shall consider a plasma with only two ion species. We shall use subscripts 1, 2 to denote the ion species (with $m_1 > m_2$) and subscripts e to denote electrons. The undisturbed plasma is assumed electrically neutral and has number densities $n_{1,\infty}$, $n_{2,\infty}$, and $n_{e,\infty}$ with $n_{1,\infty} + n_{2,\infty} = n_{e,\infty}$. The parameter α, the mixture ratio of the plasma, is defined by

$$\alpha \equiv n_{2,\infty}/n_{e,\infty}, \quad 0 \leq \alpha \leq 1 \tag{2.1}$$

We shall construct a general theory which in the limit of either $\alpha = 0$ or $\alpha = 1$ reduces to the single ion species results.

Following Lam and Greenblatt[2,3] but treating the heavy and the light ions as separate fluids, the governing equations are

$$\lambda_D^2\nabla^2\chi = \alpha N_2 + (1-\alpha)N_1 - N_e \tag{2.2}$$

$$\left.\begin{aligned} \partial N_i/\partial t + \nabla\cdot(N_i\bar{v}_i) &= 0 \quad (2.3)\\ M_i^2(\partial\bar{v}_i/\partial t + \bar{v}_i\cdot\nabla\bar{v}_i) &= \nabla\chi \quad (2.4)\end{aligned}\right\} i = 1,2$$

$$N_e = e^{-\chi} \tag{2.5}$$

where

$$\lambda_D = (kT_e/4\pi e^2N_{e,\infty})^{1/2}/L, \quad \chi = -e\phi/kT_e$$

$$\left.\begin{aligned} N_i &= n_i/n_{i,\infty}, \quad \bar{v}_i = \bar{q}_i/U_\infty \\ M_i &= U_\infty/a_i, \quad a_i^2 = kT_e/m_i \end{aligned}\right\} i = 1,2$$

$$N_e = n_e/n_{e,\infty}$$

and L is the characteristic length of the problem, ϕ is the electrostatic potential, and $\bar{q}_i$ is the dimensional ion velocity. All lengths have been nondimensionalized by L, and time t has been nondimensionalized by L/U_∞. Other symbols are standard ones. Equations (2.2) and (2.3) are "exact" within the framework of a Vlasov plasma. Equations (2.4) are the ion momentum equations under the "cold-ion" approximation ($T_1 \ll T_e$, $T_2 \ll T_e$). Equation (2.5) is the so-called Boltzmann electron density relation. For a more detailed discussion on these assumptions, see Lam and Greenblatt[2,3] and Lam.[4] Equations (2.2–2.5) are six unknowns, N_1, N_2, N_e, χ, $\bar{v}_1$, and $\bar{v}_2$. The classical two-stream instability theory[5] is contained in this formulation and is readily recovered by a linearized analysis. We shall, however, attempt to study the general nonlinear system for the satellite-ionosphere interaction problem.

III. Quasi-Neutral Approximation

The characteristic length L shall be chosen to represent the linear scale of the satellite on which the spectrometer is mounted, and is usually of the order of a meter. Except at extremely high altitudes where the ambient electron density is low, the Debye length is quite small and thus the parameter λ_D is a reasonably small number. Invoking the quasi-neutral approximation ($\lambda_D^2 \ll 1$) and using Eq. (2.5) for N_e, Eq. (2.2) yields

$$\alpha N_2 + (1-\alpha)N_1 = e^{-\chi} + 0(\lambda_D^2) \tag{3.1}$$

which is assumed valid outside thin electrostatic sheaths adjacent to the satellite surface.

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IV. Hypersonic Approximation for the Heavy Ions

To proceed further, we shall take advantage of the fact that the heavy ions of interest are usually oxygen and nitrogen ions whereas the light ions are usually hydrogen and helium. Using reasonable estimates of the electron temperature T_e, the value of the heavy ion Mach number (based on the ion-accoustic speed) is reasonably large ($M_1^2 \gg 1$) whereas the light ions Mach number is generally of order unity [$M_2^2 = 0(1)$]. In the limit of $M_1^2 \rightarrow \infty$, the heavy ions are unaffected by the electrostatic forces present in the flowfield; thus its number density N_1 remains undisturbed everywhere except in the "shadow" region of the satellite body. Confining our attention outside this shadow region, Eq. (3.1) becomes

$$N_2 = (1/\alpha)[e^{-\chi} - (1 - \alpha)] \tag{4.1}$$

where we have replaced N_1 by unity. Equation (4.1) now explicitly relates N_2 with the local electrostatic potential χ.

The assumption that the heavy ions are hypersonic results in great simplifications in the analysis, as we shall see presently. Let us multiply the light ions momentum equation, Eq. (2.4), by N_2 as given previously. We have

$$M_2^2 N_2\, Dv_2/Dt = N_2 \nabla\chi = -\nabla P \tag{4.2}$$

where

$$P = P(N_2) \equiv -\int^{\chi} N_2\, d\chi = \int^{N_2} - N_2\, d\chi \Big/ dN_2\, dN_2 \tag{4.3}$$

The light ions continuity equation, Eq. (2.3) and Eqs. (4.2) and (4.3) now describe the behavior of the quasi-neutral light-ion flowfield. By analogy with classical gas dynamics, it behaves as a compressible fluid with Eqs. (4.1) and (4.3) playing the role of the "isentropic relation," relating P and N_2. The local dimensional speed of sound a is then given by

$$a^2 \equiv \frac{kT_e}{m_2}\frac{dP}{dN_2} = -\frac{kT_e}{m_2} N_2 \frac{d\chi}{dN_2} = [1 - (1 - \alpha)e^{\chi}]\frac{kT_e}{m_2} \tag{4.4}$$

We see that when $\alpha = 1$, i.e., when no heavy ions are present, Eq. (4.3) reduces to the isothermal compressible fluid analogy stated by Lam and Greenblatt. When $\alpha < 1$, we see that the only change is that the light ion flow will have a variable local speed of sound. Also, Eq. (2.4) can be integrated once to yield, for steady flow problems,

$$\chi = (M_2^2/2)(v_2^2 - 1) \tag{4.5}$$

which is the Bernoulli's equation for the light ions. Thus we see that the faster the light ion flows, the larger χ becomes, and the lower the local value of a—which is the same qualitative behavior exhibited by classical gas dynamics (with $\gamma > 1$).

Thus, the proper Mach number for the flow of the light ions is no longer M_2. According to Eq. (4.4) the freestream "speed of sound" is now

$$a_\infty^2 = a^2(\chi = 0) = \alpha k T_e/m_2, \quad (M_1^2 = \infty)$$

Hence the proper Mach number M^* which governs the over-all characteristics of the light-ion flow is (for $M_1^2 \rightarrow \infty$),

$$M^* = U_\infty/a_\infty = (\alpha)^{-1/2} M_2 \tag{4.6}$$

If $M^* > 1$, the light-ion flow is supersonic; if $M^* < 1$, the light-ion flow is subsonic. Since $\alpha < 1$, we see that a light ion flow with $M_2 < 1$ can exhibit supersonic behavior with a sufficiently small α.

V. Subsonic Case, $M^* < 1$

When $M^* < 1$, the light-ion flow is subsonic, and solutions are quite difficult to construct due to the elliptic nature of the governing equations. However, it is possible to compute directly the light-ion flux to the forward "nose" of an arbitrary body without detailed knowledge of the flowfield. The forward "nose" of a body shall be referred to as the "ram" position.

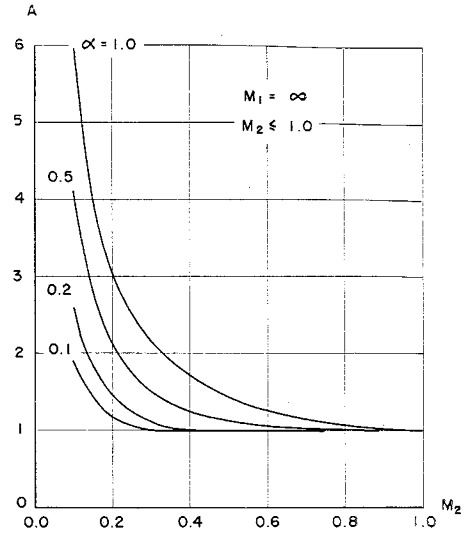

Fig. 1 Subsonic augmentation factor, $A(M_2;\ \alpha)$.

Adjacent to any absorbing surface, a detailed study of the structure of the (non-quasi-neutral) sheath shows that the light ions must enter the sheath with a normal velocity component equal to or greater than the local speed of sound. This is known as the generalized Bohm's criterion. At the ram position, we can assume that the light-ion velocity vector is normal to the sheath. Since the flow is basically subsonic, we can assume that the local light-ion velocity is sonic. Using subscript s to denote conditions at the sheath edge, the value of $(q_2^2)_s$ must equal to $(a^2)_s$. Hence, from Eq. (4.4), we have

$$(v_2^2)_s = (1/M_2^2)[1 - (1 - \alpha)e^{\chi_s}] \tag{5.1}$$

Evaluating Eq. (4.5) at the sheath edge and eliminating $(v_2^2)_s$, we have

$$2\chi_s + M_2^2 = 1 - (1 - \alpha)e^{\chi_s} \tag{5.2}$$

which relates χ_s at ram as a function of M_2 and α. The light-ion ram current is then (in dimensional form) given by

$$J_2(\text{ram}) = (n_2 q_2)_s = \alpha n_{e,\infty} U_\infty (N_2 v_2)_s = n_{e,\infty}(kT_e/m_2)^{1/2} e^{-\chi_s}[1 - (1 - \alpha)e^{\chi_s}]^{3/2} \tag{5.3}$$

Since the heavy ion is hypersonic, the heavy-ion ram current can be computed by the scoop formula, Eq. (1.2)

$$J_1(\text{ram}) = \alpha n_{e,\infty} U_\infty (1 - \alpha) \tag{5.4}$$

The ratio of J_2 to J_1 at ram is then given by

$$(J_2/J_1)_{\text{ram}} = [\alpha/(1 - \alpha)]A(M_2,\alpha) \tag{5.5}$$

where

$$A = (1/M_2\alpha)e^{-\chi_s}\,[1 - (1 - \alpha)e^{\chi_s}]^{3/2} \tag{5.6}$$

with $\chi_s(M_2,\alpha)$ given by Eq. (5.2). The function A shall be called the "augmentation factor," and is presented in Fig. 1.

VI. Supersonic Case, $M^* > 1$

When $M^* \geq 1$, the flow is generally supersonic, and the method of characteristics can readily apply (provided that a continuous solution exists, see Lam[4]). It is well known that the basic characteristics relations are solutions to the simple Prandtl-Meyer fan problem. We shall thus address ourselves to the latter problem, i.e., a steady supersonic flow about a semi-infinite flat plate at some arbitrary angle of attack.

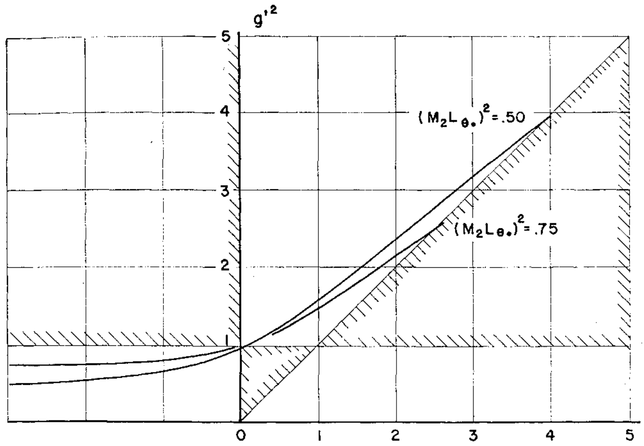

Fig. 2 Structure of the interaction layer, g'^2 vs y.

For the moment, let us suspend the hypersonic approximation for the heavy ions and formulate the Prandtl-Meyer fan problem exactly (within the quasi-neutral framework). Since both $\bar{v}_1$ and $\bar{v}_2$ can be shown to be irrotational, we can write

$$\bar{v}_1 = \nabla[rH(\theta)], \quad \bar{v}_2 = \nabla[rL(\theta)] \tag{6.1}$$

where (r,θ) are polar coordinates and $rH(\theta)$ and $rL(\theta)$ are velocity potentials for the heavy and the light ions, respectively. The governing equations are Eqs. (3.1, 2.3, and 2.4) which can be written as follows:

$$\alpha N_2 + (1-\alpha)N_1 = e^{-\chi} \tag{3.1a}$$

$$L_{\theta\theta} + L + L_\theta(\ln N_2)_\theta = 0 \tag{6.2}$$

$$H_{\theta\theta} + H + H_\theta(\ln N_1)_\theta = 0 \tag{6.3}$$

$$\chi = (M_1^2/2)(H_\theta^2 + H^2 - 1) \tag{6.4}$$

$$\chi = (M_2^2/2)(L_\theta^2 + L^2 - 1) \tag{6.5}$$

For latter reference, the undisturbed flow solution is

$$H = -\cos\theta, \quad L = -\cos\theta, \quad N_1 = N_2 = 1, \quad \chi = 0 \tag{6.6}$$

and represents a uniform flow from right to left. Manipulations of Eqs. (6.2–6.5) yield a particularly useful pair of equations

$$(L_{\theta\theta} + L)R(L_\theta,H_\theta,N_1,N_2,N_e) = 0 \tag{6.7a}$$

$$(H_{\theta\theta} + H)R(L_\theta,H_\theta,N_1,N_2,N_e) = 0 \tag{6.7b}$$

where

$$R = \alpha N_2/M_2^2L_\theta^2 + (1-\alpha)N_1/M_1^2H_\theta^2 - N_e \tag{6.8}$$

Note that the undisturbed solution satisfied Eqs. (6.7) by virtue of the fact that $H_{\theta\theta} + H = 0$ and $L_{\theta\theta} + L = 0$. Using Eqs. (6.6), the factor R can be evaluated and is readily shown to be

$$R = [\alpha/M_2^2 + (1-\alpha)/M_1^2]1/\sin^2\theta - 1 \tag{6.9}$$

We see immediately that if we define a mixture Mach number M_m by

$$1/M_m^2 \equiv \alpha/M_2^2 + (1-\alpha)/M_1^2 \tag{6.10}$$

Then if $M_m^2 > 1$, the factor R will become zero at $\theta = \theta_m$ given by

$$\sin\theta_m = 1/M_m < 1 \tag{6.11}$$

Hence the undisturbed solution, Eqs. (6.6), can prevail only for $\theta < \theta_m$. For $\theta > \theta_m$, the Prandtl-Meyer fan solution is to be obtained from setting R equal to zero. For $M_m < 1$, no such "similar" solution exists.

Thus, for arbitrary M_1, M_2, and α, M_m is the proper mixture Mach number for the quasi-neutral flow. The angle $\pi - \theta_m$ is the mixture Mach angle. When $M_1^2 = \infty$, M_m reduces to M^* defined by Eq. (4.6).

In principle, the solutions H, L, N_1, N_2, and χ can be numerically obtained by integrating the system of equations since only ordinary differential equations are involved. However, it is instructive to again assume $M_1^2 \gg 1$, $M_2^2 = 0(1)$ to allow us to proceed analytically, and in the process gain some understanding of the behaviors of the solutions and the nature of the approximations. We shall assume, in addition, that α is neither small nor near unity. These cases can readily be studied by a perturbation procedure and we shall not be concerned with them here.

Let us first show that the heavy-ion density and velocity remain undisturbed outside the shadow region when $M_1^2 \gg 1$. Differentiating Eqs. (6.4) and (6.5) and eliminating χ_θ, we have

$$H_{\theta\theta} + H = (M_2^2/M_1^2)(L_\theta/H_\theta)(L_{\theta\theta} + L) \tag{6.12}$$

Eliminating $H_{\theta\theta} + H$ between Eqs. (6.3) and (6.12) yields

$$N_{1\theta}/N_1 = -(M_2^2/M_1^2)(L_\theta/H_\theta^2)(L_{\theta\theta} + L) \tag{6.13}$$

Hence, in the limit of $M_1^2 \to \infty$, Eqs. (6.12) and (6.13) yield $H \cong -\cos\theta$ and $N_1 \cong 1$.

To construct the light ion Prandtl-Meyer fan solution, we set $R = 0$. From Eqs. (2.5) and (6.8), we have

$$\alpha N_2/M_2^2L_\theta^2 + (1-\alpha)N_1/M_1^2H_\theta^2 = e^{-\chi} \tag{6.14}$$

We see that the second term on the left-hand side can be neglected if $M_1^2 \gg 1$, so long as $H_\theta^2 \neq 0$ and $M_m^2 \cong M_2^2/\alpha \geq 1$. Using Eq. (3.1) for N_2 and setting $N_1 \cong 1$, we have

$$\frac{1 - (1-\alpha)e^\chi}{M_2^2L_\theta^2} = 1 \tag{6.15}$$

Using Eq. (6.5) to eliminate χ, we have

$$\ln[(1 - M_2^2L_\theta^2)/(1-\alpha)] = (M_2^2/2)[L_\theta^2 + L^2 - 1] \tag{6.16}$$

which is a nonlinear ordinary differential equation for $L(\theta)$. The boundary condition for $L(\theta)$ is that at $\theta = \theta_m \cong \sin^{-1}(M^*)^{-1}$, the value of $L(\theta_m)$ must match the undisturbed solution given by Eq. (6.6);

$$L(\theta = \theta_m) = -\cos\theta_m = (M_2^2 - \alpha)^{1/2}/M_2 \tag{6.17}$$

Equation (6.16) has been integrated numerically for $0.1 \leq \alpha \leq 0.9$, $1.0 \leq M^* \leq 5.0$ to obtain $L(\theta)$ for $\theta_m \leq \theta \leq \pi$. It suffices to state here that for fixed value of M^*, the solutions are quite insensitive to the value of α.

Near $\theta \cong \pi$, the preceding analysis breaks down. Mathematically, the neglected term in Eq. (6.15) is $0(M_1^2H_\theta^2)^{-1}$ with $H_\theta^2 = \sin^2\theta$ given by Eq. (6.12). Hence, near $\theta \cong \pi$ this term is no longer negligible. Physically the line $\theta = \pi$ is the boundary of the shadow region; in the hypersonic ($M_1^2 \gg 1$) limit, no heavy ions can reach the interior of the shadow region, and thus for $\theta > \pi$ the value of N_1 should be zero. A "boundary layer" in the singular perturbation sense must exist at $\theta \cong \pi$ to allow N_1 to change smoothly from unity to zero.

VII. Interaction Layer

It is physically clear that when $\pi - \theta = 0(M_1^{-1})$, the heavy ions must become dynamically involved with the light ions. To study this nonlinear interaction, we define a boundary-layer variable

$$\sigma \equiv M_1(\theta - \pi) \tag{7.1}$$

and confine our attention to the region in the physical plane where $\sigma = 0(1)$. We further write

$$H = 1 - (1/M_1^2)h(\sigma) \tag{7.2}$$

$$L = L_0 + (1/M_1M_2)g(\sigma) \tag{7.3}$$

where $h(\sigma)$ and $g(\sigma)$ are $0(1)$ and L_0 is the value of $L(\theta)$ obtained previously at $\theta = \pi$.

When $\theta_m < \theta < \pi$, the solution for $H(\theta)$ is $H = -\cos\theta$. In terms of σ this is written as

$$H(\theta \cong \pi) = -\cos\theta = 1 - (1/M_1^2)\sigma^2/2 + \dots$$

Hence the boundary condition for $h(\sigma)$ as $\sigma \to -\infty$ is, to the leading order,

$$\lim_{\sigma \to -\infty} h(\sigma) = \sigma^2/2 \qquad (7.4)$$

Analogously, we require that

$$\lim_{\sigma \to -\infty} g(\sigma) = 0 \qquad (7.5)$$

so that $L(\theta)$ defined by Eq. (7.3) matches smoothly with that found in the previous section. Differentiating Eq. (7.3) with respect to θ, we have

$$L_\theta = (1/M_2)dg/d\sigma = (1/M_2)g'$$

Hence we must also require

$$\lim_{\sigma \to -\infty} g' = M_2 L_{\theta_0} \qquad (7.6)$$

where L_{θ_0} is $L_\theta(\theta = \pi)$ computed in the previous section.

Using Eq. (7.3) in Eq. (6.2), we have

$$[N_2 g']' = 0(M_1)^{-1} \cong 0 \qquad (7.7)$$

Integrating once and using Eq. (3.1) to eliminate N_2, we have

$$g'[e^{-\chi} - (1 - \alpha)N_1] = \text{const} \qquad (7.8)$$

Eliminating N_1, N_2 between Eqs. (7.7, 7.8, and 6.14), we have in terms of the new variables,

$$e^{-\chi} g'[1 - h'^2(g'^2 - 1)/(g'^2 - h'^2)] = \text{const} \qquad (7.9)$$

Similarly, Eqs. (6.4) and (6.5) now become, to the leading order,

$$\chi \cong \tfrac{1}{2}h'^2 - h \cong \tfrac{1}{2}g'^2 + (M_2^2/2)(L_0^2 - 1) \qquad (7.10)$$

From Eq. (7.10), we can solve for g' as a function of h' and h. Eliminating χ and g' from Eq. (7.9), we can obtain in principle a first-order differential equation for h that governs the behavior of h in this interaction layer.

However, we are not so much interested in the detailed structure of the interaction layer as in what comes out of this layer. For $\theta > \pi$, we expect the heavy-ion density to be zero. The main goals of the interaction layer analysis are to verify this expectation and to predict quantitatively how the light ions emerge from this layer. To these ends we proceed as follows. First we define a new variable y to replace h:

$$y = M_2^2(1 - L_0^2) - 2h \qquad (7.11)$$

Equation (7.10) can be rewritten as

$$h'^2 = g'^2 - y = y'^2/4 \qquad (7.12)$$

Eliminating y and h'^2 from Eq. (7.9) in favor of g' and y, we obtain

$$e^{-g'^2/2}\, g'^3\, [(1 + y) - g'^2]/y = M_2^3 L_{\theta_0}^3\, e^{-(M_2 L_{\theta_0})^2/2} \qquad (7.13)$$

where the constant on the right-hand side has been evaluated by noting Eqs. (7.4) and (7.6) and the fact that as $\sigma \to -\infty$, $y \to -\infty$ [see Eq. (7.11)]. Equation (7.13) gives g' as a function of y, and is shown qualitatively in Fig. 2 for various values of $M_2 L_{\theta_0}$.

If we solve for N_1 from Eq. (6.14) and express the results in terms of the new variables, we obtain, after some algebra, the following result:

$$N_1 = (y - g'^2)(1 - g'^2)/y(1 - M_2^2 L_{\theta_0}^2) \qquad (7.14)$$

Note that as $\sigma \to -\infty$, $N_1 \to 1$. Furthermore, the solution curves given by Eq. (7.13) must terminate as they intercept the line $g'^2 = y$ where N_1 becomes zero. Thus, we have shown that the heavy-ion number density drops smoothly but rapidly from unity to zero in the interaction layer.

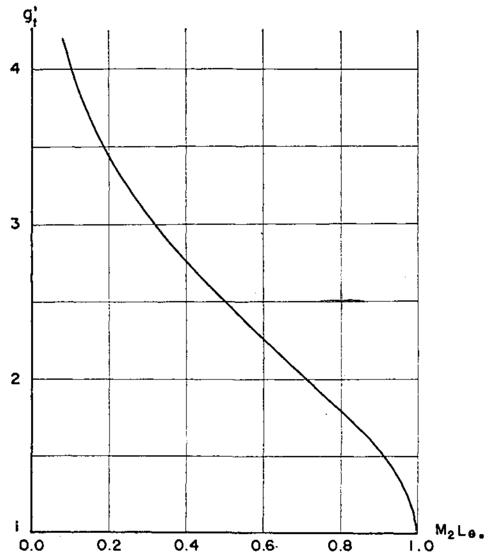

Fig. 3 Terminal value of g' vs $M_2 L_{\theta_0}$.

The θ component of the light ion velocity is proportional to g'. At the "terminal" of the interaction layer, the value of the terminal g', denoted by g'_t, can be obtained from Eq. (7.13) by setting $y = g'^2 = (g'_t)^2$. We obtain

$$g'_t e^{-\frac{1}{2}(g'_t)^2} = (M_2 L_{\theta_0})^3 e^{-(M_2 L_{\theta_0})^2/2} \qquad (7.15)$$

In Fig. 3, we have plotted the numerically obtained values of g'_t as a function of $M_2 L_{\theta_0}$.

If the actual detailed structure of the interaction layer is desired, Eq. (7.12) can be integrated once to give

$$\sigma(h) = 2 \int^{M_2^2(1-L_0^2)-2h} dy/(g'^2 - y)^{1/2} \qquad (7.16)$$

which gives σ as a function of h. Since g' is a known function of y from Eq. (7.13), this computation can readily be carried out.

VIII. Light Ions in the Shadow

We see from the previous section that the interaction layer terminates at some finite value of σ where N_1 becomes zero. As the light ions emerge from this interaction layer, they alone are responsible to maintain quasi-neutrality with the electrons.

With $N_1 = 0$ in the shadow region, Eqs. (6.7a) and (6.8) become (since $\alpha N_2 = N_e$),

$$(L_{\theta\theta} + L)[1/M_2^2 L_\theta^2 - 1] = 0 \qquad (8.1)$$

At $\theta \gtrsim \pi$, the values of L and L_θ are, to the leading order,

$$L(\theta \gtrsim \pi) \cong L(\theta \lesssim \pi) = L_0$$

$$L_\theta(\theta \gtrsim \pi) \cong 1/M_2 g'_t \qquad (8.2)$$

Since the square bracket factor is nonzero when these "initial" values are inserted, the governing equation for L in the shadow region is then simply $L_{\theta\theta} + L = 0$. The solution is then

$$L(\theta > \pi) = -B\cos(\theta - \beta) \qquad (8.3)$$

where

$$B = B(M_2, \alpha) = [L_0^2 + (1/M_2)g_t'^2]^{1/2}$$

$$\beta = \beta(M_2, \alpha) = \tan^{-1}(g'_t/M_2 L_0) \qquad (8.4)$$

Figures (4) and (5) show B and β for various values of M_2 and

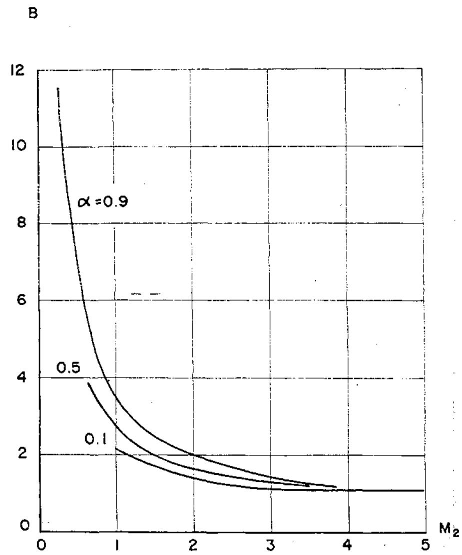

Fig. 4 Speed of light ions entering shadow region.

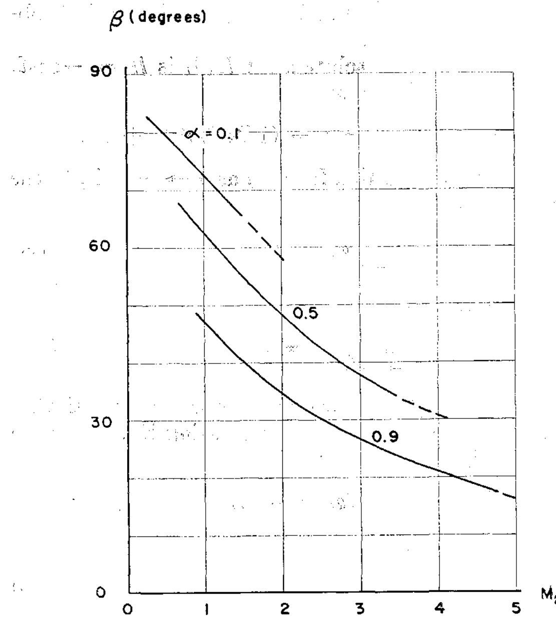

Fig. 5 Direction of light ions entering shadow region.

α. This solution represents a uniform flow with velocity B times the original oncoming flow velocity and is flowing in the $\pi + \beta$ direction. In other words, the over-all effects of the light ions expansion fan and the interaction layer are to increase the light ions flow velocity by a factor B and to turn it downward by an angle β. The heavy ions are relatively undisturbed until they reach the interaction layer where they expand rapidly to zero density. In the shadow region, the light ions flow as a single species plasma, and a new Prandtl-Meyer fan solution will take over from Eq. (8.3) when the square bracket factor in Eq. (8.1) reaches zero.

IX. Concluding Remarks

In dimensional form, the mixture speed of sound for a two-ion species quasi-neutral plasma is

$$a^2 = kT_e\{\alpha/m_2 + (1 - \alpha)/m_1\} \tag{9.1}$$

The extension to a multispecies plasma is obvious.

Let us consider a finite two-dimensional flat plate in a hypersonic heavy-ion flow, $M_1^2 \gg 1$. For simplicity we assume the plate surface is normal to the freestream. The front face of the plate will always collect heavy ions according to the simple scoop formula. In the absence of any light ions, the rear face of the plate could also collect some heavy ions, since the isothermal[2] expansion fans from the plate edges could turn ion-streamlines to strike the back of the plate.

We now introduce some light ions into the flow with $M_2 < 1$. When α is small, M^* can be quite large. Thus, the front face of the plate will collect the light ions by the scoop formula in spite of the fact that $M_2 < 1$. As α increases, the value of M^* decreases. So long as $M^* > 1$, the light ions behave supersonically. The heavy ions, however, will cease to be collected by the rear face of the plate since no heavy ions can emerge from the thin interaction layer. Thus, the rear face of the plate now collects only light ions. For larger α, M^* becomes subsonic, and now the front face of the plate collects light ions at a rate greater than the scoop formula by the augmentation factor A given in Eq. (5.6). We thus see that the ratio of number fluxes of the heavy and light ions collected by an orifice on a satellite is not related simply to the freestream mixture ratio.

There is little doubt that in the wake region of a plasma flow problem, the phenomenon of two-stream instability plays a dominant role. This is true even for single ion species plasma flows, since the focusing effect of a negatively charged body makes the wake a region of possible counter streaming. For mixture plasma flows, the light and the heavy ions counter stream whereever electric fields exist. Thus, the possibility of two-stream instability prevails over any nontrivial steady-state mixture solutions. The stability of the Prandtl-Meyer fan solutions presented here, for example, remains to be investigated.

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